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COMPUTATIONAL PROCEDURES FOR OBTAINING MODULAR NUMBERS

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DEPARTMENT OF THE ARMY
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FOR OBTAINING MODULAR NUMBERS

by

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FOREWORD

How many units should be put in each package? This question has plagued the supplier of all items that lend themselves to multi-unit packaging. Sometimes the question may be resolved in terms of the size and the shape of the resulting package, at other times in terms of decreasing the waste that may result if excess items go unused, and also in terms of reducing the necessity of handling individual units. The QM Board took the last point of view in the course of a study seeking the improvement of the packaging of supplies for combat support. More specifically, the Board defined an optimum package size of a supply item that will minimize handling effort in supplying one item to each of a number of consumers grouped into various units of known sizes. The result appeared to require extensive computations which might have limited its usefulness to situations in which high-speed computers are available. The Board, therefore, requested the Operational Mathematics Office (an element of the Cameron Station QM Activities reporting to the Director of Operations, Office of The Quartermaster General), to study the problem. The OMO developed the abbreviated computational procedures described in this report.

This is the fourth of a series of reports on the application of mathematics to QM problems. With this report the QM Board becomes the third major QM element to be assisted through reports in this series. Reports were previously prepared for other Divisions in the OQMG and for the QM Research & Engineering Center Laboratories.

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ABSTRACT

A "modular number" has been defined as an optimum package size for a supply item that will minimize handling effort in supplying one item to each a number of consumers grouped into various units of known sizes. The QM Board has suggested two different criteria for expressing numerically what is meant by "minimizing handling effort." If high-speed computing facilities are available, the determination of the modular number under either criterion could be carried out by a "brute force" technique. However, if the computations must be made manually (perhaps with the aid of a desk calculator), short-cuts are desirable. Two types of step-by-step procedures oriented toward manual computation are developed in this report. Either procedure can be adapted to either of the criteria proposed by the QM Board.

ACKNOWLEDGEMENTS

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INTRODUCTION

In connection with its Project No. 23[1] the Quartermaster Board requested the Operational Mathematics Office to develop a computational procedure for determining modular numbers. A modular number may be described as an optimum package size for a supply item that will minimize handling effort in supplying one item to each of a number of consumers grouped into units of various (known) sizes. The "consumers" referred to may be of various types, e.g., actual troops, or pieces of equipment which "consume" repair parts. Multiplying the modular number by an appropriate replacement factor gives the modular quantity, which is "that quantity of an item of supply that can be bound together or contained as a unitized element for the purpose of reducing to a minimum the number of supply handlings." For a fuller discussion of these concepts the previously cited report of the QM Board may be consulted.

In order to formulate the problem mathematically, consider an array of numbers or "frequency array" such as is exhibited in Table 1. Each number in the second column may be thought of as the size of one of the consuming units involved, while the corresponding number in the first column is the number of units of the indicated size which are to be supplied.

In order to reduce the problem to purely mathematical terms some numerical criterion of "optimization" must be agreed on. Two such criteria have been suggested by the QM Board:

I. When every number in the array is divided by the modular number (each number in the second column being regarded as present in the array the number of times indicated in the first column) the sum of the remainders obtained is less than a previously assigned tolerance T. The optimum modular number is the largest divisor which satisfies this requirement.

II. Under the same assumptions as for Criterion I, the optimum modular number is that divisor for which the sum of all the quotients and remainders is a minimum.

Only Criterion I was mentioned when the assistance of the Operational Mathematics Office (OMO) was requested in January 1960. The numerical technique referred to in this report as the "second method"

[1] Supply Segmentation and Unitization for Combat Support (Draft), Quartermaster Board, U. S. Army, Fort Lee, Virginia, November 1960.

was then developed and furnished informally to the QM Board. Criterion II was not discussed with the OMO, but was included in the Board's published report. Subsequently, the OMO developed the "first method" (placed first in this report because it would probably be preferred in the majority of cases). It was felt that a full report covering both criteria and both procedures woul be more useful than one limited to the information supplied to the QM Board.

In approaching the problem certain general observations concerning the behavior of (a) the sum of the quotients and (b) the sum of the remainders will be helpful. If the divisor is taken larger than any of the numbers in the second column of the frequency array, the quotients are all zero, while the sum of the remainders is the accumulated total. On the other hand, if the divisor is taken as 1, the remainders are all zero and the sum of the quotients is the accumulated total of the array. In other words, as the divisor decreases from a large number to 1, the sum of the quotients increases from zero to the accumulated total, while the sum of the remainders decreases from the accumulated total to zero.

The two sums also differ with regard to the effect of small changes in the divisor. It is easily seen that if the divisor is reduced in size the sum of the quotients can never decrease. If the divisor is reduced only a little the sum of the quotients may remain unchanged, but if the divisor is decreased sufficiently it will surely increase. Using mathematical terminology, the sum of the quotients is a monotonic function of the divisor.

The sum of the remainders behaves differently. Though its general tendency is downward as the divisor decreases, it fluctuates, and, in a particular instance, may move either up or down. For example, in the array of Table 1, if the divisor is taken as 137, 25 of the 42 remainders at once become zero. It might be suspected that a slightly smaller divisor would produce a larger sum of remainders. In fact, the sum of remainders is 1032 for a divisor of 137 and 1069 for a divisor of 136. The divisor must be reduced to 68 in order to obtain a smaller sum of remainders than for 137.

Under either of the two criteria previously stated, the divisor sought will be some number between 1 and S, the accumulated total of the array. if high-speed computing facilities were available, one could compute Q, or Q + R, as the case may be, for every integral divisor between 1 and S and then select the one which meets the stated conditions. In that situation this might, in fact, be the simplest procedure.

TABLE 1

GIVEN FREQUENCY ARRAY

Frequency	Number
1	185
1	231
5	106
5	183
25	137
5	163
Accumulated total	6101

However, in the absence of such facilities, this would represent a formidable computational task, and it is the purpose of this report to develop procedures suitable for manual calculation. To this end, means will be devised to reduce the amount of arithmetic required. As already indicated, two general types of methods will be described. The first of these emphasizes a systematic and orderly procedure for making the calculations, and probably would require fewer computational steps in the majority of cases. The second method seeks to "skip" as many divisors as possible (showing by various lines of reasoning that they do not need to be tested), and might be advantageous in situations where the number of trial divisors would otherwise be quite large.

If Criterion I is adopted it is possible to fix a lower limit to the divisors that need to be tested before commencing the detailed computations. If any number in the array is divided by a divisor D, the largest possible remainder that can be obtained is evidently D - 1. Thus if F denotes the total number of numbers in the set (the sum of the frequencies in Table 1), the sum of remainders R cannot exceed (D - 1)F. If we determine the largest divisor D such that

$$(D - 1)F < T,$$

or, in other words, such that

$$D < 1 + \frac{T}{F},$$

the optimum modular number under Criterion I cannot be less than this divisor, and this is the smallest divisor that needs to be tested. Another way of saying this is that the minimum divisor is the smallest integer that equals or exceeds T/F.

In applying Criterion I to the illustrative data in Table 1, we shall consider two cases: (i) T = 610 (10% of the accumulated total S) and (ii) T = 2440 (40% of S). We have F = 42, and therefore the minimum divisor is 15 (the next integer above 610/42) in case (i) and 59 (the next integer above 2440/42) in case (ii).

Under Criterion II there does not appear to be any simple means of fixing a useful minimum divisor until after some values of Q + R have been computed.

FIRST METHOD

If N_i denotes one of the numbers in the array and q_i and r_i are the quotient and remainder obtained when N_i is divided by a trial divisor D, we have the relation

$$N_i = q_i D + r_i$$

Summing over all the numbers in the array (including repetitions) gives

$$S = QD + R,$$

from which we can easily derive the relations

$$R = S - QD$$

$$Q + R = S - Q(D - 1).$$

It is clear then that it is not necessary to make separate computations of Q and R. In general, Q is easier to calculate, and either R or Q + R is then easily obtainable.

Table 2 is designed to facilitate the computation of Q for different trial divisors in a systematic and orderly manner. It should be noted that a number of different trial divisors may yield the same Q. For example, with the illustrative data given in Table 1, Q has the same value (37) for all trial divisors between 116 and 137, inclusive (as seen in Table 3). As one goes down the scale of trial divisors, Table 2 makes it easy to see at what points the value of Q changes and by how much.

The "Quotient" column contains successive numbers starting with 1. The various numbers of the array (in descending order of magnitude) are listed at the head of the columns as "dividends." Under each dividend and on the line with each quotient is entered the whole number of times that quotient divides into that dividend.

TABLE 2

DETERMINATION OF TRIAL DIVISORS FOR WHICH THE VALUE OF Q CHANGES

	Dividend					
Quotient	231	185	183	163	137	106
1	231	185	183	163	137	106
2	115	92	91	81	68	53
3	77	61	61	54	45	35
4	57	46	45	40	34	26
5	46	37	36	32	27	21
6	38	30	30	27	22	17
7	33	26	26	23	19	15
8	28	23	22	20	17	13
9	25	20	20	18	15	11
10	23	18	18	16	13	10
11	21	16	16	14	12	9
12	19	15	15	13	11	8
13	17	14	14	12	10	8
14	16	13	13	11	9	7
15	15	12	12	10	9	7
16	14	11	11	10	8	6
17	13	10	10	9	8	6
Frequency	1	1	5	5	25	5

TABLE 3

CALCULATION OF Q

D	ΔQ	Q	QD	Q + R
231	1	1	231	5871
185	1	2	370	5733
183	5	7	1281	4827
163	5	12	1956	4157
137	25	37	5069	1069
115	1	38	4370	
106	5	43	4558	
92	1	44	4048	
91	5	49	4459	
81	5	54	4374	
77	1	55	4235	
68	25	80	5440	741
61	6	86	5246	
57	1	87	4959	
54	5	92	4968	
53	5	97	5141	
46	2	99	4554	
45	30	129	5805	425
40	5	134	5360	
38	1	135	5130	
37	1	136	5032	
36	5	141	5076	
35	5	146	5110	
34	25	171	5814	458
33	1	172	5676	
32	5	177	5664	
30	6	183	5490	
28	1	184	5152	
27	30	214	5778	
26	11	225	5850	476
25	1	226	5650	
23	7	233	5359	
22	30	263	5786	
21	6	269	5649	
20	11	280	5600	
19	26	306	5814	
18	11	317	5706	
17	31	348	5916	533
16	12	360	5760	
15	37	397	5955	543
14	12	409	5726	
13	42	451		

Under Criterion I the table can be terminated when a line is about to be reached in which all the entries will be less than the minimum divisor obtained as described earlier (15 in the illustrative example). Under Criterion II one cannot be sure when to terminate the table until some of the figures in Table 3 have been calculated. For convenience the frequencies in the array of the various "dividends" are shown at the foot of the corresponding columns.

Table 3 is for the purpose of computing the values of Q and then determining the optimum modular number. The numbers "D" in the first column are the entries from the interior of Table 2, listed in decreasing numerical sequence. These are the divisors for which the value of Q changes; in other words, the value of Q for each of these divisors is different from that which would be obtained if the divisor were increased by 1.

If one of these numbers appears more than once in Table 2, it should be listed only once, but careful note should be taken of all occurrences of the number.

Opposite each D in Table 3 is entered in the second column (Δ Q) the frequency at the foot of the column of Table 2 in which D appears (or the sum of the frequencies if it appears in more than one column). These figures represent the increment in the value of Q, so that the Q column is obtained by merely accumulating the Δ Q column.

If Criterion I is adopted, the final column of Table 3 can be omitted and the table can be terminated earlier. Since R = S - QD, having R less than T (as required by Criterion I) is tantamount to having QD greater than S - T. Thus if QD is computed for each divisor before going on to the next divisor, the calculation can be stopped as soon as a value of QD larger than S - T is obtained.

In the numerical illustration two cases were considered: viz., T = 610 and T = 2440. Since S = 6101 (from Table 1), the corresponding values of S - T are 5491 and 3661, respectively. In the latter case, this value is first exceeded for the divisor 137, which is therefore the optimum modular number, and the table can be terminated at that point. In the former case, the optimum divisor is 45.

Under Criterion II the optimum divisor is the one which yields the minimum value of Q + R. Accordingly, certain values of this quantity are shown in the final column. These are computed by the formula

$$Q + R = S - QD + Q.$$

The values of Q + R are required only in those instances in which the value of QD is the largest obtained up to that point. This is because Q always increases with each new divisor, and therefore Q + R cannot be less for a given divisor than for a preceding one unless QD is larger.

The table can be terminated when a value of Q is obtained that is greater than the minimum Q + R. In the numerical example this occurs for D = 13, which yields Q = 451 (as compared with the minimum Q + R of 425). Since the minimum Q + R was obtained for the divisor 45, this is the optimum modular number under Criterion II.

SECOND METHOD

Under the second method individual quotients and remainders are computed for each number in the array for each divisor to be tested. Thus more computational steps are required to test a given trial divisor. On the other hand additional criteria are developed which make it possible to reduce substantially the number of divisors that need to be tested. Therefore this method may be advantageous in some situations. The procedures followed under Criterion I and Criterion II are sufficiently different to make it desirable to treat them separately.

1. Preliminary Analysis under Criterion I.

The first step in the procedure is the preparation of a table which summarizes the given data and serves the function of fixing certain limits between which the maximum modular number is known to fall. Table 4 is an illustration of such a table.

The first two columns of Table 4 contain the same information as Table 1, but with this difference -- that the numbers in the second column are now arranged in increasing order of magnitude.

The third column shows the total number of items in the array larger than the corresponding number in the second column, and is obtained by accumulating the numbers in the first column from the bottom up.

TABLE 4

PRELIMINARY ANALYSIS FOR FINDING MAXIMUM MODULAR NUMBER, CRITERION I

Frequency	Number	Accumulated Frequency	Minimum R	Maximum R
5	106	37	0	3885
25	137	12	530	2162
5	163	7	3955	5089
5	183	2	4770	5134
1	185	1	5685	5869
1	231		5870	5870

It is the purpose of the fourth and fifth columns of Table 4 to fix some rough limits on the values that R can have. Clearly, if D is divided into a number less than D, the remainder is the number itself. Thus the contribution to R of all the numbers in the array less than D is merely the accumulated total of all such numbers (taking into account the frequency of each, as indicated in the first column). If D itself happens to be a number of the array, its contribution to R is of course zero.

With regard to the numbers in the array greater than D, the most favorable situation would be that in which all such numbers are exactly divisible by D, and their contribution to R is zero. Thus, if one of the numbers in the second column is taken as D, a lower limit to R is the accumulated sum of the numbers less than D, and this accordingly is shown in the fourth column. Thus, the fourth column is obtained by accumulating the products of the first two columns, starting with zero in the first line and entering each cumulative total on the line below that corresponding to the last product included.

The most unfavorable situation is that in which each number in the array greater than D falls short by just one of being exactly divisible by D, so that the remainder is D - 1. Thus, if one of the numbers in the second column is taken as D, an upper limit to R would be obtained by adding to the lower limit the product of D - 1 by the number of numbers in the array greater than D (shown in the third column). This upper limit is shown in the fifth column. For example, if we take D = 137, the result is 12 (137 - 1) + 530 = 2162.

By inspection of the figures in the fourth and fifth columns, we can now fix some limits on the maximum modular number M. Evidently, there is no maximum modular number (and the problem makes no sense) if T (the preassigned tolerance for the sum of the remainder) is greater than the accumulated total of all the numbers in the array. It will be assumed hereafter that T is less than this accumulated total.[2]

If one of the numbers in the fifth column is less than T, clearly the corresponding number in the second column is a modular number, and the maximum modular number is at least equal to this quantity. <u>Thus, a lower limit to M is the largest number in the second column corresponding to a number less than T in the fifth column.</u>

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[2]If T is equal to the accumulated total, it is not difficult to see that the largest number in the array is the maximum modular number.

On the other hand, if any number in the fourth column is equal to or greater than T, then M must be less than the corresponding number in the second column. Thus, an upper limit to M is the smallest number in the second column corresponding to a number greater than or equal to T in the fourth column. If T is greater than all the numbers in the fourth column (but of course less than the accumulated total of the entire array), then it is easily seen that M is the largest number in the second column.

Let us examine Table 4 from this point of view. We shall consider two cases: (i) T = 610 (10% of the accumulated total of the array) and (ii) T = 2440 (40% of the accumulated total). In case (i) the fifth column of the table provides no lower limit to the value of M, but the fourth column indicates 163 as an upper limit. Thus we know that M is a number between 1 and 163. In case (ii) we infer that M is less than 163 but not less than 137.

2. Determination of the Maximum Modular Number under Criterion I.

In order to arrive at a systematic procedure for obtaining the exact value of M, it is convenient to consider the effect on the sum of remainders R of changing the divisor D to a smaller value D - c. Then we have

$$N = qD + r.$$

By simple algebra we obtain

$$N = q(D - c) + r + qc.$$

Thus the remainder is increased from r to r + qc provided

$$r + qc < D - c,$$

that is, provided

$$c < \frac{D - r}{q + 1}. \tag{1}$$

Thus, if c can be chosen sufficiently small so that the inequality (1) is satisfied for all numbers in the array greater than or equal to D, then all the corresponding remainders are increased. If c is also chosen so that no number in the array is between D - c and D (or equal to D - c), then the remainders corresponding to numbers less than D are the numbers themselves, and therefore remain unchanged. Thus R is increased.

We therefore take as the first trial divisor D the upper limit to M determined from the fourth column of the table. We then determine the smallest positive integer c such that the inequality (1) is not satisfied for some number in the array. As the second trial divisor we take D - c or the next smaller number preceding D in the array, whichever is the larger. It is not necessary to try the numbers in between, because the above argument shows that they will produce larger values of R than D itself.

If the second trial divisor is not a modular number, it is taken as D and the process is repeated until a modular number is reached. The first modular number so obtained is M.

It is convenient to discuss first case (ii), in which T = 2440. The calculations are shown in Table 5. The first trial divisor is 163, the upper limit to M previously determined. Note that the numbers of the array less than 163 do not need to be included in the table. By actual division we obtain the values of q and r. c is then the smallest integer that is larger than the smallest value of (D - r)/(q + 1). It is usually possible to determine by inspection which q and r give this smallest value. If all q's are equal, as is the case here, it will be the largest r. If different values of q occur, it may be necessary to examine several cases to see which remainder "goes over the top" first.

In this case it is found that D - c = 115. However, this is less than 137, which is the next smaller number preceding 163 in the array. But 137 is already known to be a modular number, since the maximum R, as shown in the final column of Table 4, is less than 2440. Therefore M = 137.

Turning now to case (i), the upper limit is again 163, and by the same reasoning as in case (ii) we conclude that the second trial divisor is 137. In this case, however, we do not know in advance whether 137 is a modular number, and the q's and r's must be computed. These are shown in the third and fourth columns of Table 6. Accumulation of the products of the remainders and the corresponding frequencies gives R = 1032, as shown in the table. As this exceeds T = 610, 137 is not a modular number and it is necessary to try further divisors.

The smallest integer that is larger than $(137 - r)/(q + 1)$ for any q and r is 22, and the next trial divisor is 137 - 22 = 115. Computation from the remainders shown in Table 6 for D = 115 gives R = 1731 -- even larger than before.

The formula would give 92 as the next trial divisor, but since the next smaller number in the array is 106, this takes precedence. The resulting value of R is 1543, which is still too large. By the formula the next trial divisor would be 92 (as before), but at this point we can take a short cut and eliminate three trial divisors.

To see how this is done look at Table 7, which shows, for D = 106, the individual contributions to R of the different numbers in the array. We note that the contribution of the number 137 alone exceeds the limit of 610. Thus, there is no possibility of R getting below this value unless the remainder for 137 is reduced. Thus the formula for c can be applied to this q and r alone, ignoring the other numbers of the array. This gives c = 38, and the next trial divisor is 106 - 38 = 68. As seen in Table 6, the resulting value of R is 661, which is still slightly too large.

Application of the formula now gives c = 7, and the next trial divisor is 68 - 7 = 61, which gives R = 855, as shown. Now, looking at Table 7, we see that, for D = 61, the first three contributions together exceed 610, so that at least one of the first three remainders must be reduced in order to get below this limit. Thus we can apply the rule to the q's and r's for the first three numbers alone, ignoring the rest of the array. This gives c = 7, and the next trial divisor is 54, which gives R = 1133.

At this point, without actually exhibiting all the individual contributions to R, we see that the contribution of the number 137 is 25 X 29 = 725, which alone is greater than 610. Thus we can, as before, apply the formula to this q and r alone, which gives c = 9, and the next trial divisor is 45. Since this gives R = 296, we see that 45 is the maximum modular number.

3. Preliminary Analysis under Criterion II.

If Criterion II is adopted, a slightly different type of preliminary analysis is appropriate. This is given in Table 8. It will be noted that the third and fifth columns of Table 4 do not appear in Table 8;

TABLE 5

DETERMINATION OF MAXIMUM MODULAR NUMBER FOR T = 2440.

Frequency	Number	D = 163 q	 r
5	163	1	0
5	183	1	20
1	185	1	22
1	231	1	68
		R = 4145	

c is smallest integer greater than $\frac{D - r}{q + 1} = \frac{163 - 68}{2}$ which is 48. D - c = 115. M = 137.

TABLE 6

DETERMINATION OF MAXIMUM MODULAR NUMBER FOR T = 610.

Frequency	Number	D = 137		D = 115		D = 106		D = 68		D = 61		D = 54		D = 45	
		q	r	q	r	q	r	q	r	q	r	q	r	q	r
5	106					1	0	1	38	1	45	1	52	2	16
25	137	1	0	1	22	1	31	2	1	2	15	2	29	3	2
5	163	1	26	1	48	1	57	2	27	2	41	3	1	3	28
5	183	1	46	1	68	1	77	2	47	3	0	3	21	4	3
1	185	1	48	1	70	1	79	2	49	3	2	3	23	4	5
1	231	1	94	2	1	2	19	3	27	3	48	4	15	5	6
		R = 1032		R = 1731		R = 1543		R = 661		R = 855		R = 1133		R = 296	

TABLE 7

INDIVIDUAL CONTRIBUTIONS TO R FOR SELECTED TRIAL DIVISORS

Frequency	Number	D = 106	D = 61
5	106	0	225
25	137	775	375
5	163	285	205
5	183	385	0
1	185	79	2
1	231	19	48
	TOTAL	1543	855

however, three new columns have been added. It is advisable for the present purpose to compute the sum of the quotients (denoted by Q) and the sum of the remainders when each number in the second column is taken as divisor. Further, the sum of remainders is subdivided to show (a) the contribution by numbers less than the divisor and (b) the contribution by numbers greater than the divisor. (There is, of course, no contribution by the divisor itself.) These separate contributions will be denoted by R_1 and R_2, respectively, and $R = R_1 + R_2$. For completeness we have added the bottom line showing what happens if a divisor greater then any of the numbers in the set is used. It is convenient to take this divisor as 1 more than the largest number otherwise present in the second column.

As regards the computation of the figures in the table we note that R_1 is the same as "Minimum R" in Table 4. After Q has been computed, we have $R_2 = S - QD - R_1$.

It is easily seen that R_1 has the same kind of behavior as Q, but in the opposite direction. Thus, if R_1 has a certain value for a given divisor D, then for any smaller D its value must be the same or less -- never greater. Therefore, the fluctuations in R are confined to R_2. (This is not apparent from Table 8, as it does not show the results when numbers other than those in the second column are taken as divisors.)

4. Determination of the Optimum Modular Number under Criterion II.

As a first step we note the smallest value of Q + R (last column) shown in the table. We know that Q + R can be made at least as small as this value, and it is not necessary to consider any trial divisors that can be shown to produce larger values. Now we look at the R_1 column to see if it contains any values greater than the smallest value of Q + R. If so, we note the smallest such value, and the corresponding number in the second column is an upper bound to the modular number, since any divisor larger than this number must necessarily give a value of Q + R larger than the smallest value shown in the table. This upper bound is taken as the starting point for obtaining further trial divisors.

It will be noted that, for the bottom number in the second column, both Q and R_2 are necessarily zero, and therefore $Q + R = R_1$. From the monotonic behavior of R_1 it follows that failure to find a value of R_1

TABLE 8

PRELIMINARY ANALYSIS FOR FINDING MODULAR NUMBER, CRITERION II

Frequency	Number	Q	R_1	R_2	Q + R
5	106	43	0	1543	1586
25	137	37	530	502	1069
5	163	12	3955	190	4157
5	183	7	4770	50	4827
1	185	2	5685	46	5733
1	231	1	5870	0	5871
0	232	0	6101	0	6101

greater than the smallest value of Q + R can occur only when the last value of Q + R is also the smallest. In such a case, the largest number actually in the set (the second from the bottom in the second column) may be taken as the starting point for trial divisors.

In Table 8 the smallest value of Q + R is 1069, and the smallest value of R_1 which exceeds 1069 is 3955, corresponding to the number 163 in the second column. Therefore the determination of trial divisors starts with 163.

On the other hand, if the table shows a value of Q greater than the smallest value of Q + R shown, the number in the second column corresponding to this value of Q is a <u>lower</u> bound to the modular number. No trial divisors less than this lower bound need be considered. In the example illustrated in Table 8, no such lower bound is obtained at this stage.

In order to find successive trial divisors we can proceed in exactly the same manner as under Criterion I (making use of formula (1)), since decreasing the trial divisor cannot decrease Q, and any reduction in the value of Q + R must result from a decrease in R. The only difference in the computations is that Q, as well as R, must be computed for each trial divisor. As soon as a trial divisor is reached for which the value of Q exceeds the value of Q + R for some previous divisor, the computations can be terminated, since any smaller divisor would produce a value of Q + R at least as large as this value of Q. The divisor, out of all those tried, which yields the smallest Q + R is the optimum modular number.

In the numerical example illustrated in Table 8, the computations parallel closely those for case (i) under Criterion I. Table 9 summarizes the results obtained for the successive trial divisors. Since there is no preassigned tolerance T, the yardstick to be applied in deciding whether certain trial divisors can be skipped is the smallest value of R previously obtained. Consider, for example, the situation just after the computations for D = 40 have been made. Table 10 shows the individual contributions to R by the different numbers in the array. The smallest previous value of R is 296 (for D = 45), and it is unnecessary to consider any trial divisor which is known in advance to produce a larger value of R. Now, it is apparent from Table 10 that the contribution of the number 137 alone exceeds 296. Thus the formula for c can be applied to this q and r only. This gives c = 6, and the next trial divisor is 40 - 6 = 34.

As indicated in Table 9, the computations are terminated with D = 13, since Q then becomes larger than the smallest value of Q + R obtained.

TABLE 9

SUMMARY OF COMPUTATIONS UNDER CRITERION II

Trial Divisor	Q	R	Q + R
163	12	4145	4157
137	37	1032	1069
115	38	1731	1769
106	43	1543	1586
68	80	661	741
61	86	855	941
54	92	1133	1225
45	129	296	425
40	134	741	875
34	171	287	458
33	172	425	597
32	177	437	614
27	214	323	537
26	225	251	476
25	226	451	677
22	263	315	578
21	269	452	721
19	306	287	593
18	317	395	712
17	348	185	533
16	360	341	701
15	397	146	543
14	409	375	784
13	451	238	689

TABLE 10

INDIVIDUAL CONTRIBUTIONS TO R FOR D = 40

Frequency	Number	q	r	Contributions to R
5	106	2	26	130
25	137	3	17	425
5	163	4	3	15
5	183	4	23	115
1	185	4	25	25
1	231	5	31	31
			TOTAL	741

Any smaller trial divisors would necessarily yield values of Q + R in excess of this smallest value. Therefore the optimum modular number under Criterion II is 45, which gives Q + R = 425.

GENERALIZATION OF THE PROBLEM

Under Criterion I, a generalized version of this problem has been suggested, in which the set of numbers is divided into a number of subsets, and the additional condition is imposed that the sum of the remainders within each subset is less than a designated number. This more general problem can be solved by successive applications of the procedure described above.

Suppose there are n subsets, numbered sequentially from 1 to n, and, in order to simplify the mathematical description of the problem, let us consider the overall set as the (n + 1)th set. For the i-th set let T_i denote the limit which the sum of the remainders must not exceed. For each set considered separately we determine by the procedure described the maximum modular number M_i.

Let M' denote the smallest of these n + 1 numbers. In many cases M' will be a modular number for all the sets. In any case, test it to see if it satisfies all the conditions. If M' is not a modular number for some set, determine the next smaller number which <u>is</u> a modular number for that set. Call this M", and test it against the remaining sets. By continuing this process a number will eventually be reached which is the maximum modular number for all the n + 1 sets. Such a number must exist, since 1 is a modular number for all sets.

CONCLUSION

If a high-speed computer were available, it would probably be more advantageous (as suggested in the QM Board report previsouly cited) to determine the modular number by trying all divisors, rather than to use the more complex procedures described in this report. The use of the "brute force" technique would greatly simplify the programming, and probably would increase computer time only marginally. The methods presented here would, however, be useful in situations where a determination must be made manually, or with a desk calculator. Even with a computer, techniques described herein might well be used to establish an upper and a lower bound, between which all possible divisors would be tested.

APPENDIX 1

CORRESPONDENCE

QUARTERMASTER BOARD
UNITED STATES ARMY
FORT LEE, VIRGINIA

IN REPLY
REFER TO: QMB

9 March 1960

SUBJECT: Procedure for Obtaining the Maximum Modular Number for a Frequency Array

TO: The Quartermaster General
Department of the Army
Washington 25, D. C.
ATTN: Dr. T. N. E. Greville
Operational Math Branch
R&E Division

1. This correspondence is in reference to informal letter of 5 February 60 from Dr. T. N. E. Greville to Mr. John Goodman of the Quartermaster Board conveying an inclosure, Procedure for Obtaining the Maximum Modular Number for a Frequency Array.

2. Mr. Goodman reports that this material has been of considerable value in promoting a combat development study, Packaging and Containerization. The Board wishes to take this opportunity to express officially its appreciation for this and other contributions to its work that have been made, both formally and informally, by the Operational Mathematics Branch.

Talbert J. Martin
Col. QMC
for HERBERT H. RASCHE
Colonel, QMC
Acting President

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Unclassified report

A "modular number" has been defined as an optimum package size for a supply item that will minimize handling effort in supplying one item to each a number of consumers grouped into various units of known sizes. The QM Board has suggested two different criteria for expressing numerically what is meant by "minimizing handling effort." If high-speed computing facilities are available, the determination of the modular number under either (over)

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1. Packaging - Optimum package size
2. Numerical Methods and Procedures - Maxima and Minima of Discrete Functions

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II. Title: Minimizing Loose Issue
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